

## ARTICLE

WILEY

# The mod $k$ chromatic index of graphs is $O(k)$

Fábio Botler<sup>1</sup>  | Lucas Colucci<sup>2</sup>  | Yoshiharu Kohayakawa<sup>2</sup>

<sup>1</sup>Programa de Engenharia de Sistemas e Computação, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

<sup>2</sup>Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

## Correspondence

Lucas Colucci, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo 05508-090, Brazil.  
Email: [lucas.colucci.souza@gmail.com](mailto:lucas.colucci.souza@gmail.com)

## Funding information

Fundação de Amparo à Pesquisa do Estado de São Paulo; Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro; Conselho Nacional de Desenvolvimento Científico e Tecnológico

## Abstract

Let  $\chi'_k(G)$  denote the minimum number of colors needed to color the edges of a graph  $G$  in a way that the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod  $k$ ). Scott proved that  $\chi'_k(G) \leq 5k^2 \log k$ , and thus settled a question of Pyber, who had asked whether  $\chi'_k(G)$  can be bounded solely as a function of  $k$ . We prove that  $\chi'_k(G) = O(k)$ , answering affirmatively a question of Scott.

## KEYWORDS

edge colorings, modulo

Throughout this paper, unless stated otherwise,  $k \geq 2$  denotes an integer. All graphs considered here are simple, and  $e(G)$  denotes the number of edges in the graph  $G$ . A  $\chi'_k$ -coloring of  $G$  is a coloring of the edges of  $G$  in which the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod  $k$ ), and we denote by  $\chi'_k(G)$  the minimum number of colors in a  $\chi'_k$ -coloring of  $G$ . Pyber [3] proved that  $\chi'_2(G) \leq 4$  for every graph  $G$  and asked whether  $\chi'_k(G)$  is bounded by some function of  $k$  only. Scott [4] proved that  $\chi'_k(G) \leq 5k^2 \log k$  for every graph  $G$ , and in turn asked if  $\chi'_k(G)$  is in fact bounded by a linear function of  $k$ . This would be best possible apart from the multiplicative constant, as  $\chi'_k(K_{1,k}) = k$ . In this paper, we answer Scott's question affirmatively.

We shall make use of the following two results.

**Lemma 1** (Mader [2]). *If  $k \geq 1$ ,  $G$  is a graph on  $n$  vertices, and  $e(G) \geq 2kn$ , then  $G$  contains a  $k$ -connected subgraph.*

**Lemma 2** (Thomassen [5]). *If  $k \geq 1$  and  $G$  is a  $(12k - 7)$ -edge-connected graph with an even number of vertices, then  $G$  has a spanning subgraph in which each vertex has degree congruent to  $k$  (mod  $2k$ ).*

We say that a graph  $G$  is  $k$ -divisible if  $k$  divides the degree of every vertex of  $G$ . Lemma 2 thus guarantees the existence of a  $k$ -divisible spanning subgraph in  $G$  when  $G$  is  $(12k - 7)$ -edge-connected.

**Lemma 3.** *If  $G$  is a graph on  $n$  vertices and does not contain a nonempty  $k$ -divisible subgraph, then  $e(G) < 2(12k - 6)n$ .*

*Proof.* Let  $G$  be a graph and suppose that  $e(G) \geq 2(12k - 6)n$ . Lemma 1 tells us that  $G$  contains a  $(12k - 6)$ -connected subgraph  $H$ . If  $H$  has an odd number of vertices, let  $H' = H - v$  for an arbitrary vertex  $v \in V(H)$ . Otherwise, let  $H' = H$ . Then  $H'$  has an even number of vertices and is  $(12k - 7)$ -connected, and hence is  $(12k - 7)$ -edge-connected also. Lemma 2 tells us that  $H'$  contains a nonempty  $k$ -divisible subgraph and therefore so does  $G$ .  $\square$

Given an integer  $d$ , we say that a graph  $G$  is  $d$ -degenerate if there is an ordering  $v_1, \dots, v_n$  of its vertices for which the number of neighbors of  $v_i$  in  $V_i = \{v_1, \dots, v_{i-1}\}$  is at most  $d$ . In this case, the neighbors of  $v_i$  in  $V_i$  are its *left neighbors* and the neighbors of  $v_i$  in  $\{v_{i+1}, \dots, v_n\}$  are its *right neighbors*. An edge  $uv$  is a *left edge* of  $v$  if  $u$  is a left neighbor of  $v$ . A left edge  $uv$  of  $v$  is a *right edge* of  $u$ .

**Lemma 4.** *Let  $G$  be a  $d$ -degenerate graph. Then  $\chi'_k(G) \leq 4d + 2k - 2$ .*

*Proof.* Let  $V(G) = \{v_1, \dots, v_n\}$  be an ordering of  $V(G)$  as above. In what follows, we color  $G$  by coloring the right edges of  $v_i$  for each  $i \in \{1, \dots, n - 1\}$  in turn, so that at each step we have a  $\chi'_k$ -coloring of the graph spanned by the right edges of  $v_1, \dots, v_i$  with the following properties: for each  $1 \leq j \leq n$ , each of the colored, left edges of  $v_j$  is colored with a distinct color, and the colors used on the right edges of  $v_j$  are distinct from the colors used on its left edges. We proceed by induction on  $i$ . Let  $i \in \{1, \dots, n - 1\}$ , and suppose that, for every  $j < i$ , the right edges of  $v_j$  are colored as above. This implies that all the left edges of  $v_i$  are colored and no right edge of  $v_i$  is colored. In what follows, we color the set  $S$  of right edges of  $v_i$  while keeping the properties of the partial coloring.

Since  $G$  is  $d$ -degenerate,  $v_i$  has at most  $d$  left edges, and hence we have at least  $3d + 2k - 2$  colors to use on its right edges. We partition these colors arbitrarily into sets  $A$  and  $B$  so that  $|A| = d + k$  and  $|B| \geq 2d + k - 2$ . Let  $j > i$  and suppose  $v_j$  is a (right) neighbor of  $v_i$ . Note that at most  $d - 1$  left edges of  $v_j$  are colored. We say that a color  $c$  is *forbidden* at  $v_j$  if a left edge of  $v_j$  is colored with  $c$ , and we call the colors in  $A$  that are not forbidden at  $v_j$  *available* at  $v_j$ .

Let  $S^*$  be a maximal subset of  $S$  that can be colored with colors in  $A$  in a way that (a) each right edge  $v_i v_j \in S^*$  is colored with a color available at  $v_j$ , and (b) the number of edges in  $S^*$  colored with any given color is congruent to  $1 \pmod k$ . Let  $\bar{S} = S \setminus S^*$  be the set of the remaining edges in  $S$ . We claim that  $|\bar{S}| < |A|$ . Assume for a contradiction that  $|\bar{S}| \geq |A|$ . For each edge  $e = v_i v_j \in \bar{S}$ , let  $A_e$  be the set of colors available at  $v_j$ , and for each color  $x \in A$ , let  $\bar{S}_x$  be the set of edges  $e$  in  $\bar{S}$  for which  $x \in A_e$ . Note that  $\sum_{e \in \bar{S}} |A_e| = \sum_{x \in A} |\bar{S}_x|$  and that  $|A_e| \geq |A| - (d - 1) = d + k - d + 1 = k + 1$  for every  $e \in \bar{S}$ . Therefore

$$(k + 1)|A| \leq (k + 1)|\bar{S}| \leq \sum_{e \in \bar{S}} |A_e| = \sum_{x \in A} |\bar{S}_x| \leq |A| \max\{|\bar{S}_x| : x \in A\},$$

whence  $\max\{|\bar{S}_x| : x \in A\} \geq k + 1$ . Let  $z \in A$  be such that  $|\bar{S}_z| = \max\{|\bar{S}_x| : x \in A\}$ . If some edge in  $S^*$  is colored with  $z$ , then we color  $k$  edges in  $\bar{S}$  with color  $z$ . If no edge in  $S^*$  is colored with  $z$ , then we color  $k + 1$  edges in  $\bar{S}$  with color  $z$ . In both cases we obtain a contradiction to the maximality of  $S^*$ . This shows that, indeed,  $|\bar{S}| < |A| = d + k$ .

Finally, we color the edges of  $\bar{S}$  consecutively and with distinct colors in  $B$ . This is possible, since for each  $v_i w \in \bar{S}$  there are at most  $d - 1 + |\bar{S}| - 1 \leq 2d + k - 3 < |B|$  colors of  $B$  that are either forbidden at  $w$ , or were used on previous edges of  $\bar{S}$ .  $\square$

Our main result is a consequence of Lemmas 3 and 4.

**Theorem 5.** *For every graph  $G$  we have  $\chi'_k(G) \leq 198k - 101$ .*

*Proof.* Let  $H$  be a maximal subgraph of  $G$  for which  $\deg_H(v) \equiv 1 \pmod{k}$  for every  $v \in V(H)$ , and let  $G' = G \setminus E(H)$ . Note that if there were an edge  $e$  joining two vertices in  $V(G) \setminus V(H)$ , then  $H' = H + e$  would be a graph for which  $\deg_{H'}(v) \equiv 1 \pmod{k}$ , contradicting the maximality of  $H$ . Therefore,  $V(G) \setminus V(H)$  is independent. Similarly, every vertex in  $V(H)$  has at most  $k - 1$  neighbors in  $V(G) \setminus V(H)$ , and  $G'[V(H)]$  has no nonempty  $k$ -divisible subgraph. By Lemma 3, every  $J \subseteq G'[V(H)]$  has less than  $2(12k - 6)|V(J)|$  edges, and hence its minimum degree is less than  $48k - 24$ . This implies that every subgraph of  $G'$  has a vertex of degree at most  $49k - 25$ , and hence  $G'$  is  $(49k - 25)$ -degenerate. Lemma 4 tells us that  $G'$  has a  $\chi'_k$ -coloring with at most  $198k - 102$  colors. We then color  $E(H)$  with a new color, and the result follows.  $\square$

Alon, Friedland, and Kalai [1] proved that when  $k$  is a prime power, the bound  $2(12k - 6)n$  in Lemma 3 can be replaced by  $(k - 1)n + 1$ . That result, together with Lemma 4, implies that  $\chi'_k(G) \leq 14k - 9$  for any  $G$  and any prime power  $k$ . In fact, those authors conjectured that the bound  $(k - 1)n + 1$  holds for any  $k$ , and if that is confirmed the improved bound on  $\chi'_k(G)$  would follow for every  $k$ .

Our arguments can be tweaked to give slightly better multiplicative constants, but we do not think it is worth pursuing this because we think we would be far from the truth still. Although we are not able to offer any strong evidence, we conjecture the following.

**Conjecture 6.** *There is a constant  $C$  such that  $\chi'_k(G) \leq k + C$  for every graph  $G$ .*

We know that  $C$  in the conjecture above has to be at least 2, because one can prove that the graph  $G$  obtained from a  $K_{k,k}$  by adding a universal vertex satisfies  $\chi'_k(G) = k + 2$ .

## ACKNOWLEDGMENTS

This study has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil - CAPES - Finance Code 001. Fábio Botler is supported by CNPq (423395/2018-1), by FAPERJ (211.305/2019 and 201.334/2022), and by UFRJ (ALV 23.733). Lucas Colucci is supported by FAPESP (2020/08252-2). Yoshiharu Kohayakawa is partially supported by CNPq (311412/2018-1, 423833/2018-9, and 406248/2021-4) and FAPESP (2018/04876-1 and 2019/13364-7). The research that led to this paper started at WoPOCA 2019, which was financed by FAPESP (2015/11937-9) and CNPq (425340/2016-3 and 423833/2018-9). FAPERJ is the Rio de Janeiro Research Foundation. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

**ORCID**

Fábio Botler  <http://orcid.org/0000-0003-2028-199X>

Lucas Colucci  <http://orcid.org/0000-0002-7390-8314>

**REFERENCES**

1. N. Alon, S. Friedland, and G. Kalai, *Regular subgraphs of almost regular graphs*, J. Combin. Theory Ser. B. **37** (1984), no. 1, 79–91.
2. W. Mader, *Existenz  $n$ -fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte*, Abh. Math. Semin. Univ. Hambg. **37** (1972), 86–97.
3. L. Pyber, *Covering the edges of a graph by ...*, Sets, Graphs and Numbers (Budapest, 1991), vol. **60**, Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1992, pp. 583–610.
4. A. D. Scott, *On graph decompositions modulo  $k$* , Discrete Math. **175** (1997), no. 1–3, 289–291.
5. C. Thomassen, *Graph factors modulo  $k$* , J. Combin. Theory Ser. B. **106** (2014), 174–177.

**How to cite this article:** F. Botler, L. Colucci, and Y. Kohayakawa, *The mod  $k$  chromatic index of graphs is  $O(k)$* , J. Graph Theory. 2023;102:197–200.  
<https://doi.org/10.1002/jgt.22866>